Quantum Jump Method for General Time-Local Master Equations

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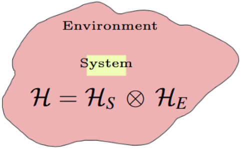
Introduction

Lindblad equation for the state operator ρ_t of a system

$$\frac{d}{dt}\rho_t = -\, i[H, \rho_t] + \sum_k \Gamma_{k,t} \left(L_k \rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\} \right)$$

where $\Gamma_{k,t} \geq 0$



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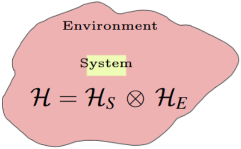

Unravel by stochastic evolution for the state vector of the system ψ_t. The Monte Carlo average E reconstructs the system state

$$\rho_t = \mathrm{E}(|\psi_t\rangle\langle\psi_t|)$$

The stochastic evolution of ψ_t can either a **quantum jump** or **quantum diffusion** process.

Introduction

Simple quantum trajectory method for **general time local master equations**

$$\cancel{\Gamma_{k,t} \geq 0}$$

For example: Redfield/TCL master equations, Gaussian models.

Earlier approaches

- Expanding the Hilbert space (Breuer et al., Phys. Rev. A **59** (1999))
- Explicitly taking memory effects into account (Piilo et al., Phys. Rev. Lett. **100** (2008))

Introduction

What does our method has to add:

- **Time local** stochastic jump process
- In the **Hilbert space of the system**
- Introduce an auxiliary process for a scalar μ_t we call the **Influence-Martingale**

Influence-Martingale method

A stochastic process for the state vector ψ_t and an **enslaved process** μ_t

The solution of the master equation is then reconstructed by

$$\rho_t = \mathsf{E}(\mu_t \, |\psi_t\rangle\langle\psi_t|)$$

The only constraint of the method is that during the time horizon $[0, \tau]$

$$\int_0^\tau dt \, |\Gamma_{k,t}| < \infty$$

Lindblad case

For simplicity, let us consider a qubit interacting with a zero temperature bath.

$$\frac{d}{dt}\rho_t = -i\omega[\sigma_z, \rho_t] - \Gamma\left(\sigma_-\rho_t\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_t\}\right)$$

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A possible **unravelling**

$$d\psi_t = -i\omega\sigma_z\psi_t\, dt - \frac{\Gamma}{2}\left(\sigma_+\sigma_- - \|\sigma_-\psi_t\|^2\right)\psi_t\, dt - \left(\frac{\sigma_-\psi_t}{\|\sigma_-\psi_t\|} - \psi_t\right) dN$$

N is a **Poisson process**, its increment takes values $dN = 0, 1$ and

$$\mathrm{E}(dN|\psi_t) = \Gamma\,\|\sigma_-\psi_t\|^2 dt$$

How to reconstruct the Lindblad Equation?

Remember $\rho_t = \mathrm{E}(|\psi_t\rangle\langle\psi_t|)$, compute the differential

$$d(|\psi_t\rangle\langle\psi_t|) = (d|\psi_t\rangle)\langle\psi_t| + |\psi_t\rangle(d\langle\psi_t|) + (d|\psi_t\rangle)(d\langle\psi_t|)$$

using the rules of stochastic calculus:

$\times$	dt	dN
dt	0	0
dN	0	dN

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$$d(|\psi_t\rangle\langle\psi_t|) = -i\omega[\sigma_z, |\psi_t\rangle\langle\psi_t|]dt + \sigma_-|\psi_t\rangle\langle\psi_t|\sigma_+ \frac{dN}{\|\sigma_-\psi_t\|^2}$$
$$- \frac{\Gamma}{2}\{\sigma_+\sigma_-, |\psi_t\rangle\langle\psi_t|\}dt + (\Gamma\|\sigma_-\psi_t\|^2 dt - dN)|\psi_t\rangle\langle\psi_t|$$

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using the rules of stochastic calculus:

$\times$	dt	dN
dt	0	0
dN	0	dN

$$E(d(|\psi_t\rangle\langle\psi_t|)) = -i\omega[\sigma_z, \rho_t]dt + \Gamma\sigma_-\rho_t\sigma_+ dt - \frac{\Gamma}{2}\{\sigma_+\sigma_-, \rho_t\}dt + \cancel{(\Gamma\|\sigma_-\psi_t\|^2 dt - dN)}|\psi_t\rangle\langle\psi_t|$$

Influence Martingale method

We now study the same master equation, but Γ_t can take **negative values**

$$\frac{d}{dt}\rho_t = -i\omega[\sigma_z, \rho_t] - \Gamma_t \left(\sigma_- \rho_t \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_t\} \right)$$

The system state satisfies the jump process

$$d\psi = -i\omega\sigma_z\psi_t \, dt - \frac{\Gamma_t}{2}\left(\sigma_+\sigma_- - \|\sigma_-\psi_t\|^2\right)\psi_t \, dt - \left(\frac{\sigma_-\psi_t}{\|\sigma_-\psi_t\|} - \psi_t \right) dN_t$$

N is a Poisson process, its increment takes values $dN = 0$, 1 and

$$\mathrm{E}(dN_t|\psi_t) = |\Gamma_t| \, \|\sigma_-\psi_t\|^2 dt$$

and

$$d\mu_t = \mu_t(\mathrm{sign}(\Gamma_t) - 1)(dN_t - |\Gamma_t| \|\sigma_-\psi_t\| dt)$$

Influence Martingale method

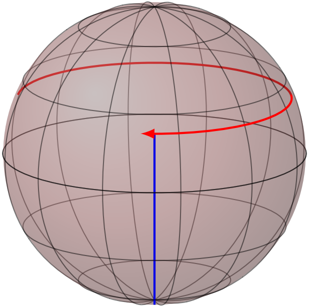
Some remarks

- μ_t is a martingale since

$$E(d\mu_t|\psi_t) = \mu_t(\mathrm{sign}(\Gamma_t) - 1)(E(dN_t|\psi_t) - |\Gamma_t|\|\sigma_-\psi_t\|dt) = 0$$

- μ_t can take **negative values**
- The state jump process gives

$$d(\|\psi_t\|^2) = (dN_t - \Gamma\|\sigma_-\psi_t\|dt)(1 - \|\psi_t\|^2)$$



Reconstructing the master equation

Remember $\rho_t = \mathsf{E}(\mu_t|\psi_t\rangle\langle\psi_t|)$

$$d(\mu_t|\psi_t\rangle\langle\psi_t|) = (d\mu_t)|\psi_t\rangle\langle\psi_t| + \mu_t\, d(|\psi_t\rangle\langle\psi_t|) + (d\mu_t)d(|\psi_t\rangle\langle\psi_t|)$$

Reconstructing the master equation

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Similar to the Lindblad case, we find that

$$\begin{aligned} d(|\psi_t\rangle\langle\psi_t|) = &-i\omega[\sigma_z, |\psi_t\rangle\langle\psi_t|]dt + \sigma_-|\psi_t\rangle\langle\psi_t|\sigma_+ \frac{dN_t}{\|\sigma_-\psi_t\|^2} \\ &- \frac{\Gamma_t}{2}\{\sigma_+\sigma_-, |\psi_t\rangle\langle\psi_t|\}dt + (\Gamma_t\|\sigma_-\psi_t\|^2 dt - dN_t)|\psi_t\rangle\langle\psi_t| \end{aligned}$$

Reconstructing the master equation

Remember $\rho_t = \mathsf{E}(\mu_t|\psi_t\rangle\langle\psi_t|)$

$$d(\mu_t|\psi_t\rangle\langle\psi_t|) = (d\mu_t)|\psi_t\rangle\langle\psi_t| + \mu_t\, d(|\psi_t\rangle\langle\psi_t|) + (d\mu_t)d(|\psi_t\rangle\langle\psi_t|)$$

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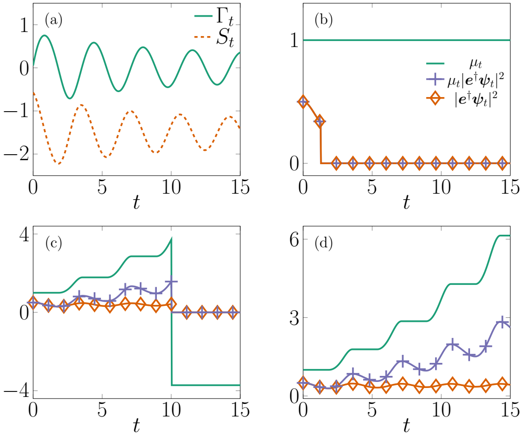
$$d(|\psi_t\rangle\langle\psi_t|) = -i\omega[\sigma_z, |\psi_t\rangle\langle\psi_t|]dt + \sigma_-|\psi_t\rangle\langle\psi_t|\sigma_+ \frac{dN_t}{\|\sigma_-\psi_t\|^2} - \frac{\Gamma_t}{2}\{\sigma_+\sigma_-, |\psi_t\rangle\langle\psi_t|\}dt + (\Gamma_t\|\sigma_-\psi_t\|^2 dt - dN_t)|\psi_t\rangle\langle\psi_t|$$

However

$$d(\mu_t|\psi_t\rangle\langle\psi_t|) = -i\omega[\sigma_z, |\psi_t\rangle\langle\psi_t|]dt + \mathbf{sign}(\mathbf{\Gamma_t})\sigma_-|\psi_t\rangle\langle\psi_t|\sigma_+ \frac{dN_t}{\|\sigma_-\psi_t\|^2} - \frac{\Gamma_t}{2}\{\sigma_+\sigma_-, |\psi_t\rangle\langle\psi_t|\}dt + (\Gamma\|\sigma_-\psi_t\|^2 dt - \mathbf{sign}(\mathbf{\Gamma_t})dN_t)|\psi_t\rangle\langle\psi_t|$$

Example: two level system in photonic band gap

$$\dot{\rho}_t = \frac{S_t}{2\imath}[\sigma_+\sigma_-, \rho_t] + \Gamma_t\left(\sigma_-\rho_t\sigma_+ - \frac{1}{2}[\sigma_+\sigma_-, \rho_t]\right)$$



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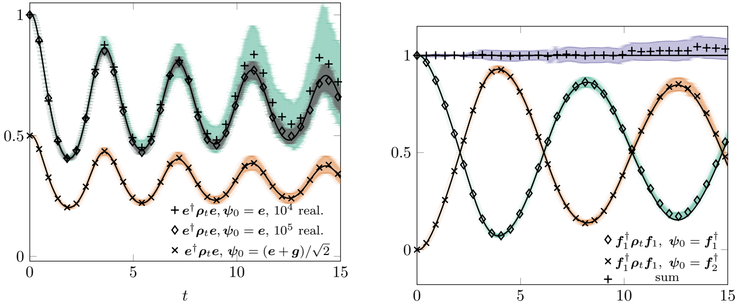

Figure: $f_1 = (e+g)/\sqrt{2}$, $f_2 = (e-g)/\sqrt{2}$

General Influence Martingale Method

$$\frac{d}{dt}\rho_t = -i[H,\rho_t] + \sum_k \Gamma_{k,t}\left(L_k\rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\}\right)$$

The jump process is

$$d\psi_t = -\, iH\psi_t dt - \sum_k \Gamma_{k,t}\frac{L_k^\dagger L_k - \|L_k\psi_t\|^2}{2}\psi_t dt + \sum_k \left(\frac{L_k\psi_t}{\|L_k\psi_t\|} - \psi_t\right) dN_k$$

with

$$dN_k\, dN_l = \delta_{k,l} dN_k, \quad \mathrm{E}(dN_k|\psi_t) = |\Gamma_{k,t}|\|L_k\psi_t\|^2 dt$$

and

$$d\mu_t = \mu_t \sum_k (\mathrm{sign}(\Gamma_{k,t}) - 1)(dN_k - |\Gamma_{k,t}|\|L_k\psi_t\|^2 dt)$$

Redfield equation for two non-interacting qubits

$$H = \sum_{i=1}^{2} \omega^{(i)} \sigma_+^{(i)} \sigma_-^{(i)} + \sum_{k=1}^{N} \left(\epsilon_k \, b_k^\dagger b_k + \sum_{i=1}^{2} g_k (\sigma_+^{(i)} \, \mathrm{b}_k + \mathrm{b}_k^\dagger \, \sigma_-^{(i)}) \right)$$

The Redfield equation in the interaction picture is

$$\frac{d}{dt} \rho_t = - \int_0^\infty ds \, \mathrm{tr}_B [H_I(t), [H_I(s), \rho_t \otimes \rho_B]]$$

This leads to a general time local master equation with two jump operators L_1 and L_2 which satisfy

$$[L_k, L_l] = 0, \quad [L_k, L_l^\dagger] = \delta_{k,l}$$

and rates $\boldsymbol{\Gamma_1 < 0}$ and $\Gamma_2 > 0$.

Redfield equation for two non-interacting qubits

We consider basis states

$$|g\rangle, \quad |w_1\rangle = L_1|g\rangle, \quad |w_2\rangle = L_2|g\rangle, \quad |w_3\rangle = L_1L_2|g\rangle$$

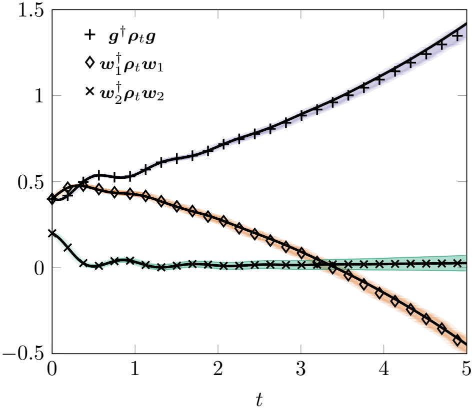

Figure: $\psi_0 = \sqrt{0.7}|g\rangle + \sqrt{0.2}|w_1\rangle + \sqrt{0.1}|w_2\rangle$

Even More General Influence Martingale Method

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \sum_k \Gamma_{k,t} \left(L_k \rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\} \right)$$

The state jump process is

$$d\psi_t = - iH\psi_t dt - \sum_k \Gamma_{k,t} \frac{L_k^\dagger L_k - \|L_k\psi_t\|^2}{2}\psi_t dt + \sum_k \left(\frac{L_k\psi_t}{\|L_k\psi_t\|} - \psi_t \right) dN_k$$

with

$$dN_k \, dN_l = \delta_{k,l} dN_k, \quad \mathrm{E}(dN_k|\psi_t) = \gamma_{k,t}\|L_k\psi_t\|^2 dt$$

and

$$d\mu_t = \mu_t \sum_k g_{k,t}(dN_k - \gamma_{k,t}\|L_k\psi_t\|^2 dt)$$

Constraint

$$\Gamma_{k,t} = (1 + g_{k,t})\gamma_{k,t}$$

Conclusion

- Simple quantum jump method for time local master equations
- Possible applications: Simulation of large quantum systems, Thermodynamics, measurement picture, ...

Thank you for your attention!