Liouvillian Exceptional Points in a Quantum Thermal Machine

arXiv:2101.11553

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EPs are points of coalescence of eigenvalues and eigenvectors of a non-Hermitian matrix.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2} \quad \Rightarrow \quad \lambda_1 = \lambda_2;\ v_1 = v_2 \text{ at } (a - d)^2 + 4bc = 0$$

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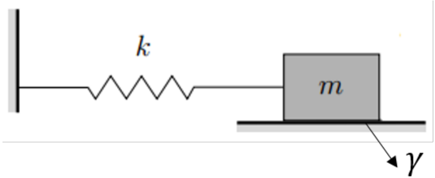
Non-Hermitian matrices are non-diagonalisable at EPs, Jordan form can be constructed.

$$S^{\;1} M S = \underbrace{\begin{pmatrix} {}_1 & 0 \\ 0 & {}_2 \end{pmatrix}}_{\text{non-EP}} \qquad T^{\;1} M_{\text{EP}} T = \underbrace{\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}}_{\text{EP}}$$

EP in a damped harmonic oscillator



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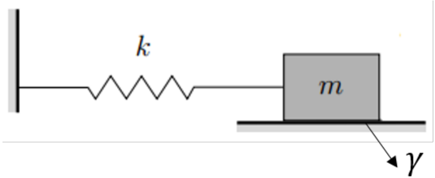

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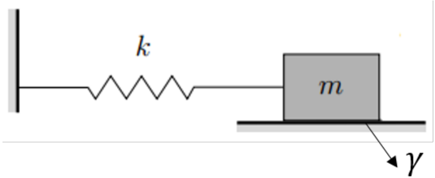

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Underdamping: $\quad ^2 < mk$

Overdamping: $\quad ^2 > mk$

Critical damping: $\quad ^2 = mk$

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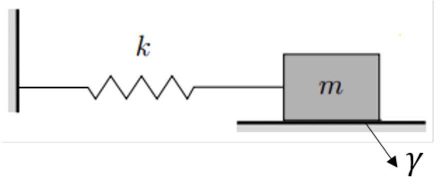

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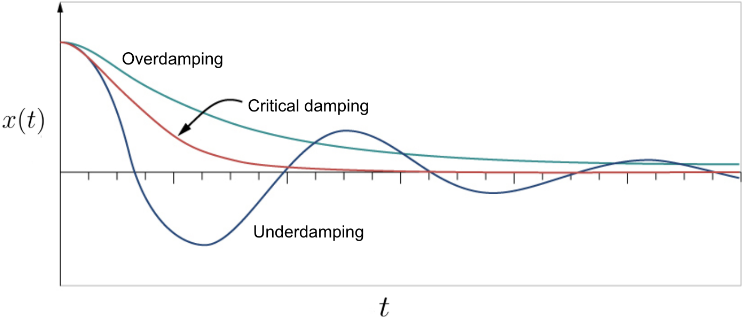

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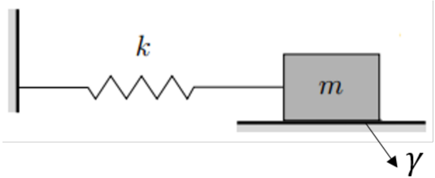

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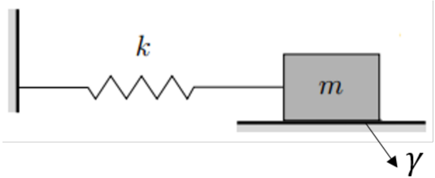

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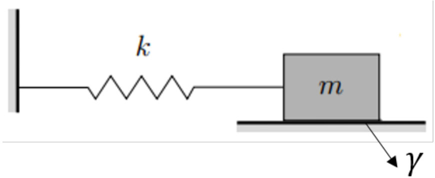

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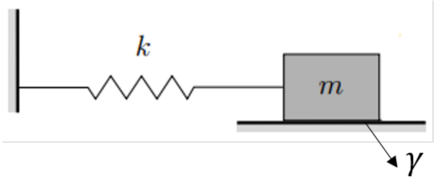

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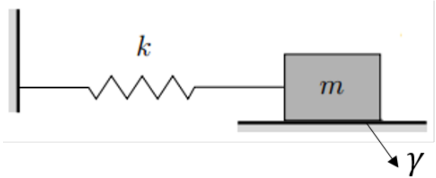

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$$\begin{pmatrix} p \\ x \end{pmatrix} = c_1 e^{\;t} v + c_2(1 + t)e^{\;t} v^{0}$$

Exceptional points in practice

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Sensing

Hodaei et al. Nature **548**, 187 (2017)

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Quantum dynamics

Uzdin et al. PRA **88**, 022505 (2013)

Koslo et al. Entropy **19**, 136 (2017)

Minganti et al. PRA **100**, 062131 (2019)

Comparison to previous works

Previous works:

- Semi-classical approach without quantum jumps (Naghiloo et al., Nat. Phys. **15**, 1232{1236 (2019))
- Single system thermalising with a reservoir (Hatano, Molecular Phys. **117**, 2121 (2019)).
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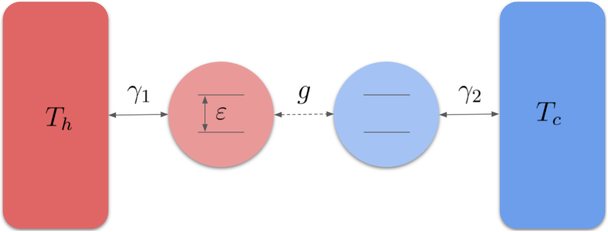
- Understanding of dynamics missing.

In this work:

- Fully quantum approach with Liouvillian.
- Coupled systems in an out-of-equilibrium situation.
- Signatures in long and short time dynamics.
- Critical damping in a quantum system.

Our model: two-qubit entanglement machine



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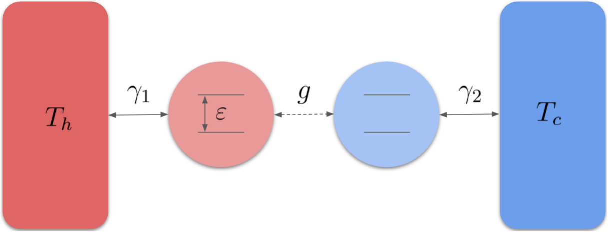

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Steady state entanglement can be created [Brask et al. 2015].

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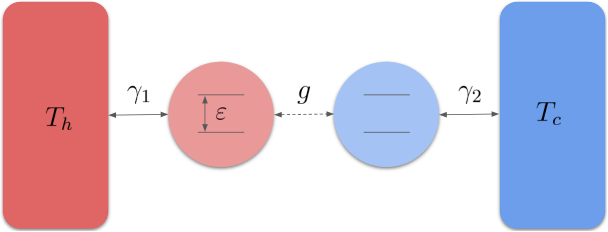

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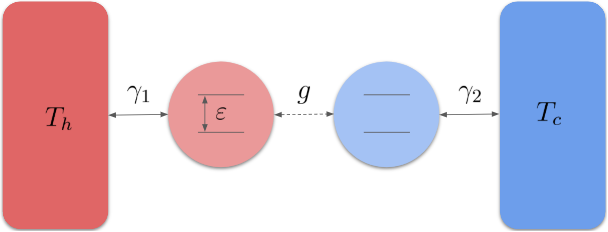

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Steady state entanglement can be created [Brask et al. 2015].
Heat current affects entanglement production [SK et al. 2020].

$$\dot{}(t) = L\,(t) = i[H_S + H_{\text{int}}, (t)] + \sum_{j \in \{h,c\}} {}^{+}_{j} D\left[{}^{(j)}_{+}\right](t) + {}_{j} D\left[{}^{(j)}\right](t)$$

$$H_{\text{int}} = g\left({}^{(h)}_{+}\,{}^{(c)} + {}^{(h)}\,{}^{(c)}_{+} \right)$$

Non-Hermitian description: Liouvillian matrix

Vectorise and obtain a 16 16 matrix representation of L.

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- Restrict $\rho(0)$ to only have populations and coherences corresponding to $|01\rangle\langle10|$ and $|10\rangle\langle01|$.

$$\rho(0) = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & a & 0 \\ 0 & a & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}$$

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$$\tilde{L} = \begin{pmatrix} {}_1 \quad {}_2 & {}^+_2 & {}^+_1 & 0 & 0 & 0 \\ {}_2 & {}_1 \quad {}^+_2 & 0 & {}^+_1 & ig & ig \\ {}_1 & 0 & {}^+_1 \quad {}_2 & {}^+_2 & ig & ig \\ 0 & {}_1 & {}_2 & {}^+_1 \quad {}^+_2 & 0 & 0 \\ 0 & ig & ig & 0 & \frac{\Gamma}{2} & 0 \\ 0 & ig & ig & 0 & 0 & \frac{\Gamma}{2} \end{pmatrix} :$$

$$_1 = 0; \quad _2 = \quad ;$$

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$$_4 = \frac{1}{2} \quad ;$$

$$_{5;6} = \frac{1}{2}$$

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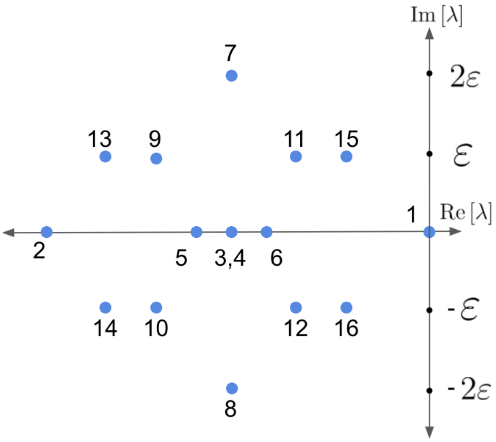
$$_j = {}_j^{+} + {}_j; \quad = \sum_j \; _j$$



Liouvillian spectrum and exceptional points

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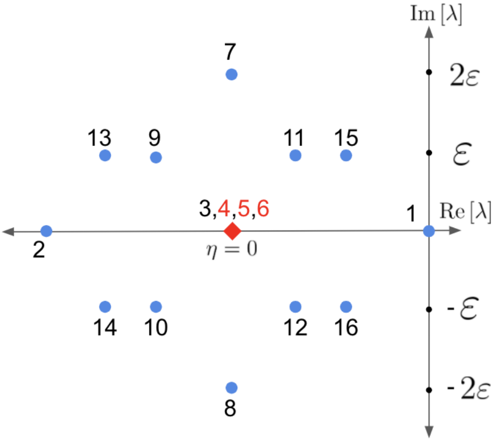

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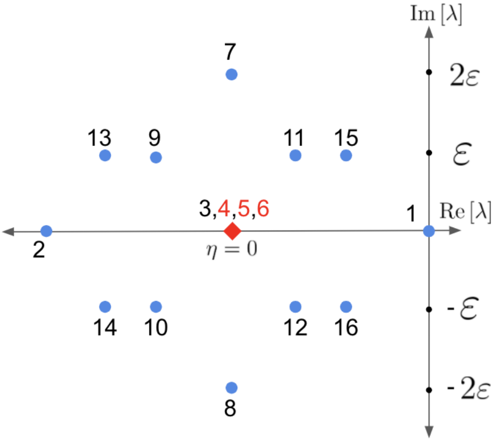

Exceptional points do not leave signatures in the steady state [Minganti et al. 2019]

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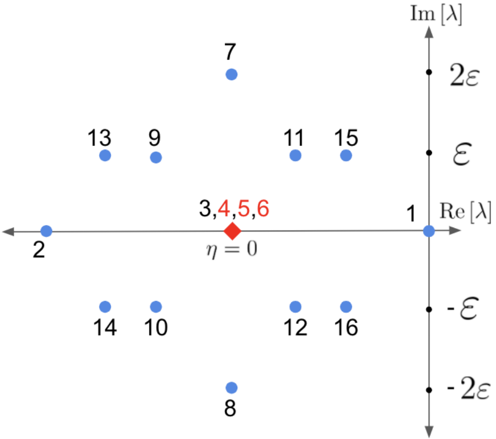

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We must look at the time-dynamics!

What does the EP imply?

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Harmonic oscillator-like behaviour!

Exact dynamics

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Right eigenmatrices: $L\ \ _i = \ \ _i\ \ _i$ Left eigenmatrices: $^y_i L = \ \ _i\ \ ^y_i$

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At the $= 0$ EP, ${}_4 = \; {}_5 = \; {}_6 =$

$$(t) = \; {}_{ss} + \sum_{i=2,3} c_i e^{\; {}_i t} \; {}_i + \; c_4 + \; c_5 t + \; c_6 \frac{t^2}{2} \; e^{\; t} \; {}_4 + \; c_5 + \; c_6 t \; e^{\; t} \; {}^{0} + c_6 e^{\; t} \; {}^{00}.$$

^{0} and ^{00} are generalised eigenvectors.

$(\tilde{L} \;) \; {}_4 = 0$, $(\tilde{L} \;) \; {}^{0} = \; {}_4$, $(\tilde{L} \;) \; {}^{00} = \; {}^{0}$

Long-time dynamics: critical damping

Goal: to see how the system approaches the steady state (how close it is to the steady state at long times).

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$$R(t) = \frac{T(\;(t);\; _{ss})}{T(\;(t);\; _{ss})}$$

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$$R(t) = \frac{T(\; (t); \; _{ss})}{T(\; (t); \; _{ss})} \qquad \frac{c_4 + c_5 t + c_6 \frac{t^2}{2} \quad _4 + (c_5 + c_6 t) \quad ^0 + c_6 \quad ^{00} \quad _1}{|| c_4 \; _4 + c_5 e^{\; t} \; _5 + c_6 e^{\; t} \; _6 ||_1}$$

At long times, $R(t) \quad O(t^2)/O(e^{\; t}) < 1$

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At long times, $R(t) \quad \frac{O(t^2)}{O(e^{\ t})} < 1$

As $t \to \infty$, $R(t) \to 0$.

$$_{\text{singlet}} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$_{\text{ground}} = |00\rangle\langle 00|$$

$$_{\text{thermal}} = \frac{e^{\ H/k_B T}}{\mathrm{Tr}\ e^{\ H/k_B T}}$$

Conclusions and perspective

The two qubit thermal machine shows a number of second and higher-order EPs.

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- control of many-body systems

Thank you

For more details: arXiv:2101.11553

Full Liouvillian

$$_1 = 0\;;\quad _2 = \qquad;\quad _3 = \quad \frac{1}{2}\;;\quad _4 = \quad \frac{1}{2}\;;\quad _{5;6} = \quad \frac{1}{2}$$

$$_{7;8} = \quad 2i" \quad \overline{2}$$

$$_{9;10} = \quad i" \quad \overline{2} \quad p\,\overline{\qquad} \qquad\qquad _{11;12} = \quad i" \quad \overline{2} + p\,\overline{\qquad}$$

$$_{13;14} = \quad i" \quad \overline{2} \quad p\,\overline{\quad + \quad} \qquad\qquad _{15;16} = \quad i" \quad \overline{2} + p\,\overline{\quad + \quad}$$

with the de nitions $= \frac{_1 + \; _2}{2};\quad = \frac{_1 \quad _2}{2};\quad _j = \; ^+_j + \; _j\;;\quad = p\,\overline{\quad ^2 \quad 4g^2};\quad = q\,\overline{g^2 + \frac{^2_1 + \; ^2_2}{8};\quad = \frac{1}{8} \quad 4(\qquad)^2 \quad 16g^2}$ and

$$= \; ^2_1 + \; ^2_2 + 2 \; _1 \quad _2 \quad 3 \; ^+_2 + 2 \; ^+_1 \quad ^+_2 \quad 3 \; _2$$

Exceptional points

Short-time dynamics

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Initial state engineering allows for complete control of dynamics.

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EP-characteristic polynomial terms in time can be detected at short times.